Chapter Six: Relations (and More)

6.1. Introduction: More Logical Form

1. Relations and Relation Letters. Once more we expand the formal language. Our motivation for doing so is again to ensure that the formal test of validity rightly evaluates intuitively valid arguments. The following argument, for example, strikes us as **valid**.

1. Jack's a person, and he scaled the Cathedral of Learning.
2. The Cathedral of Learning is a skyscraper.

∴ Someone scaled a skyscraper.

The best translation we can manage in the Chapter Five language is the following.

A: Jack **B**: The Cathedral of Learning
G: __ is a person **H**: __ scaled the Cathedral of Learning
I: __ is a skyscraper **J**: __ scaled a skyscraper

1. $(GA \wedge HA)$
2. IB

$\therefore \exists x (Gx \wedge Jx)$

But this formal argument is susceptible to a simple validity counterexample.

A: Jack **B**: The Cathedral of Learning
G: __ is a person **H**: __ scaled the Cathedral of Learning
I: __ is a skyscraper **J**: __ scaled a skyscraper

1. Jack's a person, and he scaled the Cathedral of Learning.	1. (GA ∧ HA)
2. The Cathedral of Learning is a skyscraper	2. IB
∴ Someone scaled a skyscraper.	∴ ∃x (Gx ∧ Jx)

𝔻: {**Jack**, **The Cathedral of Learning**}

A: **Jack** **B**: **The Cathedral of Learning**
G: {**Jack**} **H**: {**Jack**}
I:{**The Cathedral of Learning**} **J**: { }

The formal test of validity judges this argument **invalid**.

Of course, consistent with our original intuition that the argument is valid, we're liable to balk at the situation presented as a counterexample: a case where, even though the Cathedral of Learning is a skyscraper, Jack scaled the Cathedral of Learning without scaling a skyscraper. Still, it would be a poor solution here to rig the semantics to stamp this formal argument valid, the better to agree with our judgment of the English argument. For there are plenty of glaringly **invalid** arguments which take the same translation.

A: Neko **B**: Elvis
G: __ is a person **H**: __ ate fish tacos
I: __ is a gambler **J**: __ is a unicorn

1. Neko's a person, and she ate fish tacos.	1. (GA ∧ HA)
2. Elvis is a gambler.	2. IB
∴ Someone is a unicorn.	∴ ∃x (Gx ∧ Jx)

A: Lucretia **B**: Letitia
G: __ is a person **H**: __ dyed his/her hair black
I: __ is feeling optimistic **J**: __ owns a magic wand

1. Lucretia's a person, and she dyed her hair black.	1. (GA ∧ HA)
2. Letitia is feeling optimistic.	2. IB
∴ Someone owns a magic wand.	∴ ∃x (Gx ∧ Jx)

The problem with our formal translation of the original (valid) argument is a familiar one: there are significant, but neglected, overlaps among the premises and conclusion of the English argument. For example, the word "skyscraper" appears in both the second premise and the conclusion; yet the formal translation reveals no such overlap.

1. Jack's a person, and he scaled the Cathedral of Learning.	1. (GA ∧ HA)
2. The Cathedral of Learning is a **skyscraper**.	2. IB
∴ Someone scaled a **skyscraper**.	∴ ∃x (Gx ∧ **J**x)

As the above invalid arguments show, as far as the formal language is concerned the predicate letters "I" and "J" can each mean any old thing, however unrelated.

The shortcomings of the translation are ones we can't overcome within the Chapter Five formal language. We have, for instance, no way to bring out the common features of "is a skyscraper" and "scaled a skyscraper" using just predicate letters (and likewise between "The Cathedral of Learning" and "scaled The Cathedral of Learning").[1]

[1] Moreover, the predicate "scaled a skyscraper" papers over the tacit existential quantifier in "a skyscraper". (For instance, "Jack scaled a skyscraper" says there's some object, x, which is a skyscraper and which Jack scaled.)

Central to both those failings, the translation overlooks the repeated word "scaled". Note that this doesn't behave like a predicate (of the sort translated by a predicate letter in Chapter Five), such as "is a person" or "is a gambler". For such predicates each have one 'blank' to fill with a name. Putting "Jack" in those blanks, for instance, yields the (true) complete sentence "Jack is a person" and the (false) sentence "Jack is a gambler".

By contrast, adding "Jack" to "scaled" yields "Jack scaled" – which isn't a complete sentence.

Yet note: if we add a second name, "the Cathedral of Learning," we do get a complete sentence: "Jack scaled the Cathedral of Learning". Whereas a predicate such as "is a gambler" is **one** filled **blank** short of a complete sentence, "scaled" needs **two blanks** filled to yield a complete sentence.

Reserving the term "predicate" for our **one-place** sentence-makers, we call a two-place sentence-maker (such as "scaled") a **relation phrase** of English.[2]

To accommodate such two-place relation phrases, our formal language needs a counterpart not found in the language of Chapter Five. So besides our earlier one-place predicate letters we now introduce two-place **relation letters**. The same letters that counted as predicate letters will be pressed into service here – though making clear the two blanks to be filled by adding a **numerical superscript** "**2**". We will likewise mark a letter as a (mere) predicate letter by adding a superscript "**1**". Our translation keys will then look like the following example.

G^1a: a is a person
H^2ab: a scaled b

[2] Using 'phrase' very loosely here, to mean a string of words. A string such as "is the same age as" won't count as a **grammatical** phrase of English, in the sense of being a 'natural part' (a 'constituent') of an English sentence. Here again the grammars of English and the formal language diverge.

And just as the subject followed a predicate letter, both blank-filling formal terms follow a relation letter.

A: Jack **H^2ab**: a scaled b
B: The Cathedral of Learning

Jack scaled the Cathedral of Learning: **H^2AB**

2. Relation Letters Extended. Once we understand how the predicate letters of old can be extended to more than one place, we see as well that there's no need to stop at two-place relations. For English speaks naturally of **three-place relations**.

A: Albuquerque **G^3abc**: a is between b and c
B: Pittsburgh
C: San Diego

Albuquerque is between Pittsburgh and San Diego.

G^3ABC

(The "and" might tempt us to treat this sentence as a conjunction of two smaller claims. But we should resist that temptation: the claim that "Albuquerque is between Pittsburgh and San Diego" is **not** a conjunction of the two smaller nonsensical pseudo-sentences "Albuquerque is between Pittsburgh" and "Albuquerque is between San Diego".)

Further examples of three-place relations appear in the sentences "The director of the commercial replaced Kitty with Barbie" and "Neko prefers Jack to Suki".[3]

[3] In fact, once we see that relation letters can have any (integral) number of places, a further possibility suggests itself: a **zero-place relation letter**. This would be a (capital) letter requiring no terms added in order to qualify as a complete formal sentence. That describes a **sentence letter**: a letter constituting a complete formal sentence on its own, with no need for added name letters.

Viewing the sentence letters of old as zero-place relation letters, we could sweep them under the umbrella of relation letters, thereby extending relation letters from G to Z. But we won't bother to do that here – leaving capital letters P through Z as sentence letters.

These changes yield the following construction rules for the expanded formal language of Chapter Six.

Chapter Six Construction Rules

Terms:

T1. Name letters are terms

T2. Variables are terms

Atomic Formulas:

A1. Sentence letters are atomic formulas.

A2. A relation letter with ***n*** many places, followed by ***n*** many terms, is an atomic formula.

Formulas:

1. Atomic formulas are formulas.
2. If ● is a formula, then ~● is a formula.
3. If ● and ▲ are formulas, then (● ∧ ▲) is a formula.
4. If ● and ▲ are formulas, then (● ∨ ▲) is a formula.
5. If ● and ▲ are formulas, then (● → ▲) is a formula.
6. If ● and ▲ are formulas, then (● ↔ ▲) is a formula.
7. If ★ is a variable and ● is a formula, then

$$\exists ★ \bullet$$

and

$$\forall ★ \bullet$$

are both formulas.

Of course the successes of previous chapters show that this expanded language isn't always needed to demonstrate argument validity. And even when multi-place relations are discussed the simpler methods of earlier

chapters may suffice to show that an argument is valid. The following argument is one extreme example.

1. Albuquerque is between Pittsburgh and San Diego.

∴ It's not the case that Albuquerque isn't between Pittsburgh and San Diego.

Even if we sweep the entire English premise under a sentence letter, the formal argument still comes out valid.

P: Albuquerque is between Pittsburgh and San Diego.

1. P

∴ ~~P

But if wielding the full Chapter Six language upon this argument is logical overkill, the argument which began this discussion illustrates that in some cases nothing less than the expanded language of Chapter Six suffices to formally detect an argument's validity.